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THE STABLE TOPOLOGY OF 4-MANIFOLDS

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The stable theory (which allows connected sums with $S^2\times S^2$) is unified and extended using current 4-manifold techniques. Principal new results are a stable 5-dimensional s-cobordism theorem, and the fact that 1-connected smooth 4-manifold pairs stably have handle decompositions with no 1-handles.

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4-manifold   stable h-cobordism theorem   transverse sphere

1. Introduction

The goal here is to show how current techniques for studying surfaces in 4-manifolds lead to complete and efficient solutions of certain stable problems.

The first results revolve around the idea of a transverse family of 2-spheres. If a collection of 2-spheres has a framed embedded transverse family then algebraically cancelling pairs of intersections can be cancelled geometrically. This is because the transverse family can be used to construct framed embedded Whitney discs. Next there is an algebraic characterization of collections of 2-spheres in M^4 which have a framed embedded transverse family in some stabilization $M\# k(S^2\times S^2)$. Together these facts imply stable embedding and isotopy theorems for 2-spheres. The embedding theorem gives a new approach to the stable surgery of Cappell and Shaneson [1]. The isotopy theorem implies an analog of the s-cobordism theorem:

1.1. Theorem. *Suppose $(W^5; M_+, M_-)$ is a smooth s-cobordism whose boundary cobordism from ∂M_+ to ∂M_- has a product structure. Then for some k the k-fold stabilization of W by connected sum along an arc with $(S^2\times S^2)\times I$ has a product structure extending the one given on the boundary.*

Connected sum along an arc means to delete a tubular neighborhood of an arc between the ends in each cobordism, and then identify the resulting $S^3\times I$ boundary pieces. The result is an s-cobordism from $M_+\# k(S^2\times S^2)$ to $M_-\# k(S^2\times S^2)$.

Note that Theorem 1.1 implies in particular that $M_+ \# k(S^2\times S^2) \simeq M_- \# k(S^2\times S^2)$. This fact, which is quite a bit weaker than Theorem 1.1, follows from elementary handlebody theory and has been known for some time [14]. The full s-cobordism theorem (without stabilization) has been proved in some very special cases by Shaneson [10] using surgery and the 6-dimensional s-cobordism theorem. These techniques do not seem to give a proof of Theorem 1.1.

The second result concerns handlebody structures of 4-manifolds.

Generally if M^n is a compact manifold, $\partial_0 M \subset \partial M$ a codimension zero submanifold, then M can be presented as a handlebody built up from $\partial_0 M$. The geometrical connectivity question is: if $(M, \partial_0 M)$ is k-connected, is there a handle decomposition with no handles of dimension $\leqslant k$? The answer is yes if $k \leqslant n-4$, and the proof is part of the standard proof of the s-cobordism theorem (see Wall [11]). A.J. Casson has an example showing that the answer is sometimes no when $k = n-3$. The result here is that in the $k = n-3$ case the answer becomes yes if the manifold is stabilized properly.

1.2. Theorem. *Suppose $(M^n, \partial_0 M)$ is $n-3$ connected, compact and is smooth if $n = 4$ or 5. Then for some k the k-fold stabilization of $(M, \partial_0 M)$ by boundary connected sum with $(S^2\times D^{n-2}, S^2\times S^{n-3})$ has a handlebody structure with handles only in dimensions $n-2$, $n-1$, and n.*

If $n \geqslant 6$ this is proved with a fact about 2-complexes, and the s-cobordism theorem. When $n = 5$ the same proof works when Theorem 1.1 is substituted for the s-cobordism theorem. When $n = 4$ a simple 2-disc argument is used.

These stable results have been useful in several contexts. For instance both Theorems 1.1 and 1.2 are used in [8] to construct book decompositions of 5- and 6-dimensional manifolds. Perhaps more importantly the stable theory gives a simple introduction to 4-manifold techniques. Since this paper was written (1978) Freedman has made great advances [3, 6], culminating in the proof of the topological Poincaré conjecture [4]. A main ingredient of this work has been embedding theorems for towers of immersed discs. Proof of these theorems can be given using essentially only the techniques used in this paper, used relentlessly [9].

2. Transverse families

2.1. Definition. Suppose $\alpha_1, \ldots, \alpha_n$ are maps of surfaces into a 4-manifold M. Then a *transverse family* is a collection $\alpha_1^\perp, \ldots, \alpha_n^\perp : S^2 \to M$ such that the image of $\alpha_i^\perp$ intersects α_i transversely in one point, and is disjoint from α_j for $j \neq i$.

A transverse family is disjoint, framed, etc. if the $\alpha_i^\perp$ are disjoint, framed, etc.

The first technique is Norman's trick for removing intersection points [7]. If $\alpha^\perp$ is a sphere transverse to α and β a surface intersecting α, then β can be changed to be disjoint from α by connected sum with a copy of $\alpha^\perp$ (see Fig. 1).

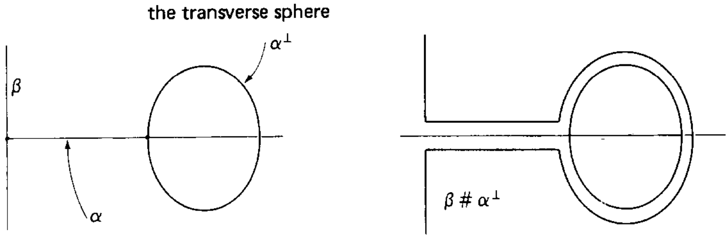

Fig. 1

Another technique is *twisting* (see [5]). Suppose a disc is embedded (in a 4-ball) with a boundary arc lying on a second disc. The first disc can be changed to approach the second disc in a twisted fashion. The picture (see Fig. 2) is of a 3-dimensional slice in which the second disc appears as a line (the line of intersection).

In the picture the new disc has a line of double points. This is resolved to an embedding by pushing out a little in the 4th dimension. There is a new intersection point with the second disc.

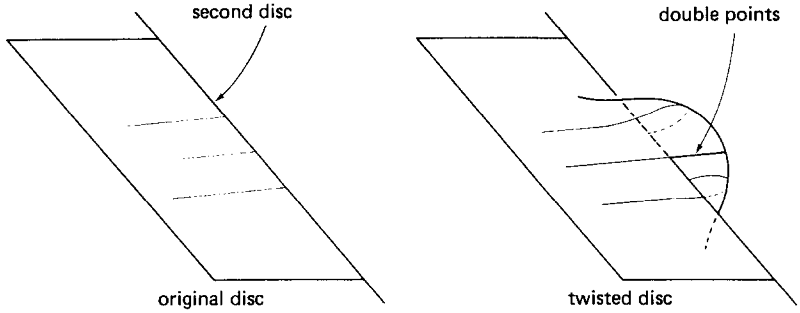

Fig. 2

The purpose of the operation is to change the framing. The normal bundle of the common arc has a natural framing. Suppose the normal bundle of the first disc has a framing which agrees with the natural one near the ends of the arc. Then there is an obstruction in $\mathbb{Z} = \pi_1 \mathrm{SO}(2)$ to matching these framings along the entire arc. The twisting operation changes the obstruction by ± 1, as can be seen directly by drawing little arrows in the picture to represent the framing. The operation can be repeated to cancel the obstruction.

These techniques are used to construct discs suitable for the classical Whitney trick.

2.2. Proposition. *Suppose $\alpha_i : S^2 \to M^4$, $i = 1, \ldots, n$ has a disjoint framed embedded transverse family $\alpha_i^{\perp} : S^2 \to M$, and suppose there is a pair of intersection (self-intersection points) of the $\{\alpha_i\}$ which algebraically cancel. Then there is an isotopy (regular homotopy) disjoint from the transverse family which reduces the number of intersection points.*

Proof. Let x_0, x_1 be the cancelling pair of intersection points. If they are joined by arcs on the two sheets to form a circle, the statement that they algebraically cancel means exactly that the surfaces define a framing of the normal bundle of the circle, and the circle is nullhomotopic.

Let $\beta: D^2 \to M$ be a nullhomotopy of the circle, in general position. Change β by connected sums with copies of the α_i to eliminate intersections with $\alpha_i^{\perp}$. Twist if necessary so that the framing of the normal bundle of β agrees with the natural framing on the boundary. Remove self-intersections by pushing part of one sheet at an intersection point across the boundary of the disc. (This introduces more intersections with the α_i.) Finally change by connected sums with copies of the $\alpha_i^{\perp}$ to eliminate intersections with the α_i. Since the $\alpha_i^{\perp}$ are disjoint framed embedded this does not introduce any new intersections with anything, and does not change the framing at the boundary of the disc.

The result is an embedded disc with interior disjoint from the $\{\alpha_i\}$ and $\{\alpha_i^{\perp}\}$, and properly framed for the Whitney isotopy (see Fig. 3).

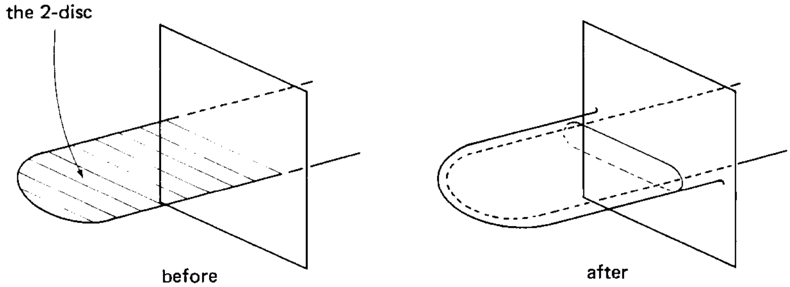

Fig. 3

This reduces the number of intersections, and since it takes place in a neighborhood of the disc it is disjoint from the transverse family.

To use this result we need criteria for the existence of such transverse families. An *algebraically transverse* family is a collection $\alpha_i^{\perp}$ with algebraic intersections $\alpha_i \cdot \alpha_j^{\perp} = 0$ if $i \neq j$, and $= 1$ if $i = j$.

2.3. Proposition. *Suppose $\alpha_1, \ldots, \alpha_n: S^2 \to M^4$. Then the following are equivalent.*

(1) *$\{\alpha_i\}$ has a framed algebraically transverse family, and $\pi_1(M - U(\alpha_i)) \to \pi_1 M$ is an isomorphism.*

(2) *$\{\alpha_i\}$ has a framed transverse family in M.*

(3) *$\{\alpha_i\}$ has a disjoint framed embedded transverse family in $M \# n(S^2 \times S^2)$.*

(Note that framed algebraically transverse families are homologically accessible since the intersections are defined on $\pi_2 M \simeq H_2(M; \mathbb{Z}[\pi_1 M])$, and an element can be represented by a framed immersion if and only if $w_2: H_2(M; \mathbb{Z}) \to \mathbb{Z}/2$ vanishes on it.)

Proof. Suppose condition (1) holds. The π_1 condition implies that the circle fiber of the normal bundle of each α_i is nullhomotopic in the complement. These nullhomotopics when glued to the normal discs give a transverse family for the $\{\alpha_i\}$, say $\delta_1, \ldots, \delta_n$. They may not be framed, however.

Let $\alpha_1^\perp, \ldots, \alpha_2^\perp$ be the given framed algebraically transverse family. Then $\alpha_i^\perp - \delta_i \in \pi_2(M) \simeq H_2(M; \mathbb{Z}\pi_1 M)$. The second homomorphism in the exact sequence

$$\cdots \to H_2(M - \bigcup \alpha_i; \mathbb{Z}\pi_1 M) \to H_2(M; \mathbb{Z}\pi_1 M) \to H_2(M, M - \bigcup \alpha_i; \mathbb{Z}\pi_1 M) \to \cdots$$

is given by intersections with the α_i, so since $(\alpha_i^\perp - \delta_i) \cdot \alpha_j = 0$, all j these elements come from $H_2(M - \bigcup \alpha_i; \mathbb{Z}\pi_1 M)$. Next the π_1 condition implies that

$$\pi_2(M - \bigcup \alpha_i) \simeq H_2(M - \bigcup \alpha_i; \mathbb{Z}\pi_1(M - \bigcup \alpha_i)) \simeq H_2(M - \bigcup \alpha_i; \mathbb{Z}_1 M),$$

so the $\alpha_i^\perp - \delta_i$ can be represented by maps $S^2 \to M - \bigcup \alpha_i$. Adding these maps to the δ_i gives a transverse family of maps representing $\alpha_i^\perp$. Approximate these by immersions. Then since the $\alpha_i^\perp$ are framed, these are a framed immersed transverse family. This shows that (1) implies (2).

An earlier version of the paper gave a direct geometric proof of this: Represent the algebraic transverse family by b_i. Then copies of the δ_j can be added to remove the extra intersections in $b_i \cap \alpha_j$. One then sees that the result of these sums is regularly homotopic to b_i, because the intersections cancel algebraically. However the regular homotopy is given by a picture, which some readers found obscure.

Next suppose $\alpha_1^\perp, \ldots, \alpha_n^\perp$ are a framed transverse family. Let a_i, b_i represent the new pairs of 2-spheres in $M \# n(S^2 \times S^2)$. Change $\alpha_i^\perp$ by connected sum with a_i, then the resulting family has a disjoint framed transverse family (the b_i) disjoint from the α_i. Therefore the new $\alpha_i^\perp$ family can be changed by connected sum with copies of the b_i to remove intersections and self intersections. This shows that (2) implies (3).

Finally (3) implies (1). The collapse $M \# n(S^2 \times S^2) \to M$ is a normal map and does not change intersections with α_i, so the image of a framed transverse family is framed transverse. The π_1 condition is satisfied if there is a transverse family. This completes the proof of Proposition 2.3.

The final ingredient in the applications is the Casson move described by Freedman [3]. This gives a way to realize the fundamental group hypothesis of Proposition 2.3. The first example implies the stable surgery of [1].

2.4. Corollary. *Suppose $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in H_2(M; \mathbb{Z}[\pi_1 M])$, $w_2\beta_i = 0$ for all i, and $\beta_i \cdot \alpha_j = 0$ if $i \neq j$, $= 1$ if $i = j$. Then $\alpha_1, \ldots, \alpha_n$ can be represented by disjoint framed embeddings in $M \# n(S^2 \times S^2)$ if and only if w_2 and the intersection and self-intersection forms (Wall [13, § 1.5]) vanish on $\{\alpha_i\}$. The embeddings obtained satisfy $\pi_1(M - \bigcup \alpha_i(S^2)) \to \pi_1 M$ is an isomorphism.*

Proof. First represent $\{\alpha_i\}$ by framed immersions. The classes β_i imply that the kernel of $\pi_1(M - \bigcup \alpha_i(S^2)) \to \pi_1 M$ is perfect, so Casson moves may be performed to make this an isomorphism. The $\{\alpha_i\}$ now satisfy the hypotheses of Proposition 2.3, so have a disjoint framed embedded transverse family in $M \# n(S^2 \times S^2)$. Since all of the intersections and self-intersections of the α_i algebraically cancel, Proposition 2.2 implies that they can be regular homotoped (in $M \# n(S^2 \times S^2)$) to be disjointly embedded.

2.5. Corollary. *Suppose $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n : S^2 \to M$ are framed immersions such that $\pi_1(M - \bigcup \alpha_i) \to \pi_1 M$ and $\pi_1(M - \bigcup \beta_i) \to \pi_1 M$ are isomorphisms, and the algebraic intersections are $\alpha_i \cdot \beta_j = 0$ if $i \neq j$, $= 1$ if $i = j$. Then there is an isotopy of $M \# 2n(S^2 \times S^2)$ which carries β_i (for each i) to a sphere disjoint from α_j, $j \neq i$, and which intersects α_i in exactly one point.*

(Another way to say this is that the α_i and β_i start out algebraically transverse, and end up geometrically transverse. The number of $S^2 \times S^2$ summonds required can be reduced to n with very little effort.)

Proof. The algebraic hypotheses imply that the kernel of $\pi_1(M - \bigcup \alpha_i - \bigcup \beta_i) \to \pi_1 M$ is perfect. Therefore there is an isotopy of M (composed of Casson moves which introduce intersections between the families) which changes $\{\beta_i\}$ to make this an isomorphism. The union $\{\alpha_i\} \cup \{\beta_i\}$ is algebraically transverse to itself, so it satisfies the hypotheses of Proposition 2.3. Therefore it has a disjoint framed embedded transverse family in $M \# (2n)(S^2 \times S^2)$. Proposition 2.2 now applies to produce an isotopy which removes algebraically cancelling intersections.

This corollary implies the s-cobordism Theorem 1.1 by just substituting it for the Whitney trick in the usual proof of the high dimensional result.

3. Geometrical connectivity

Suppose, as in Theorem 1.2, that $(M, \partial_0 M)$ is $(n-3)$-connected. Then by duality $(M, \partial_1 M)$ satisfies all known algebraic conditions to be a relative 2-complex [12]. Stabilization by $\# D^{n-2} \times S^2$ changes $(M, \partial_1 M)$ to $(M \vee S^2, \partial_1 M)$ (up to homotopy). It is well known that repeating this operation does eventually allow realization of $(M, \partial_1 M)$ as a relative 2-complex (see J. Cohen [2]).

Suppose $(M, \partial_1 M)$ is simple homotopy equivalent to a 2-complex $(K, \partial_1 M)$, and assuming $n \geq 5$ embed K in M. A regular neighborhood of K has handles only in the desired dimensions (dual to the cells of K), and the complement is an s-cobordism. If $n \geq 6$ the s-cobordism is a collar neighborhood of $\partial_0 M$, so $(M, \partial_0 M)$ has the desired handlebody structure. If $n = 5$ further stabilization of M by $\# D^{n-2} \times S^2$ stabilize the s-cobordism by sums with $S^2 \times S^2 \times I$. Theorem 1.1 therefore eventually applies to give a trivialization of the s-cobordism.

The case $n = 3$ is trivial, so $n = 4$ remains. First choose a handlebody structure with no 0-handles. Let $(N, \partial_0 N)$ be the manifold pair obtained by deleting the open

1-handles from $(M, \partial_0 M)$, and adding the new boundary to $\partial_0 N$. This pair certainly has a handlebody structure with no 1-handles. The theorem follows by showing that

$$(N, \partial_0 N) \simeq (M, \partial_0 M) \# k(S^1 \times D^3, S^1 \times S^2).$$

Since $(M, \partial_0 M)$ is 1-connected, the 1-handles are homotopic into $\partial_0 M$. These homotopies can be represented by immersed 2-discs with an arc in the boundary on $\partial_0 M$, and the rest of the boundary (the core arc of the 1-handle) in the interior. Intersection and self-intersection points can be removed by pushing them across the free part of the boundary. This yields disjointly embedded 2-discs. Disjoint tubular neighborhoods of these discs are balls which present M as a connected sum of a smaller copy of M and a bunch of balls. The 1-handles lie inside these balls, so N is a connected sum of M, and balls with a 1-handle deleted. Such a deleted ball is isomorphic to $S^1 \times D^3$, as required.

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